Eur. Phys. J. C 55, 499–507 (2008)
DOI 10.1140/epjc/s10052-008-0607-5

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Path integral treatment of pair creation by the step potential

R. Rekioua, B. Guettou, T. Boudjedaa[a]

Laboratoire de Physique Théorique, Faculté des Sciences, Université de Jijel, BP 98, Ouled Aissa, 18000 Jijel, Algérie

Received: 30 September 2007 / Revised version: 19 March 2008 /
Published online: 8 May 2008 –

Abstract. In this paper, we use a formalism including supersymmetry to calculate the Green function for the spinning relativistic particle in the presence of the step potential. This is an example of one of simple but fundamental inhomogeneous fields. We obtain an explicit expression of the Polyakov spin factor and the presence of the step gives rise to a correction of a delta potential in the spin–field coupling. The calculation is done using the perturbation technique. The corresponding positive- and negative-energy states are obtained. This Green function is used to evaluate a pair-creation probability in this potential.

PACS. 03.65.-w; 03.30.+p; 03.65.Ca; 03.65.Db; 12.38.Cy

1 Introduction

The phenomenon of pair creation by a classical electromagnetic background was first studied by Schwinger [1]. In his famous paper, the probability of the vacuum–vacuum transition in a constant electromagnetic background is calculated in terms of the imaginary part of the effective Lagrangian, and the appearance of an imaginary part in the effective action is associated to an instability of the vacuum caused by the pair creation corresponding to a quantum field effect. Recently, the phenomenon of pair creation from strong background fields, which is related to the known Klein paradox, attracted some attention in the case of various field configurations and the more realistic external and solvable case of fields, non-uniform and non-homogeneous, are important and are of great interest in the non-perturbative treatment of this phenomenon finding applications from particle physics to cosmology [2, 3]. Many different methods, based on the solution of the wave equations [33, 34], the solution of the Heisenberg equations [4] or an exact calculation of relative Green functions [1, 5, 6] have been developed to study pair creation in external fields. In the latter case, the proper time method is used to express the propagator as a Feynman path integral [7, 8]. This is also known as the spinning worldline path integral formulation. This is a supersymmetric model using commuting variables and anti-commuting variables such as Grassmann variables and attempts to combine relativistic invariance and the discrete nature of spin required by relativity. This formalism was developed from the calculation point of view in the case of concrete configurations of external fields [9–16], which are a combination of a constant field and a plane wave field. The study of time dependent and spatially inhomogeneous electric backgrounds is also of great interest in the applications. Recently, the resolvent technique [17] has been applied to a spatially inhomogeneous background. The numerical worldline techniques [18] combine worldline discretization with Monte Carlo techniques to develop an algorithm for the computation of pair-creation rates in scalar QED for inhomogeneous fields. Furthermore, different approaches such as the WKB approximation [19, 20] and instanton techniques [21] are also important in extracting information on this pair-creation phenomenon. Worldline instantaneous techniques [22, 23] are extended to the non-uniform and non-homogeneous cases by doing semiclassical non-perturbative computations presenting a more explicit result for the situation of a time dependent electric field and implicit results for the spatially inhomogeneous case.

Our aim in this paper is to use this supersymmetry formalism to calculate the Green function for the spinning relativistic particle in the presence of the step potential. This is an example of one of simple but fundamental inhomogeneous fields. Then we calculate, using this Green function, the pair-creation probability relative to the step potential. This gives us the assurance that all our first calculations are correct by extracting information on this pair creation. The method can be extended to more general solvable spatially inhomogeneous cases.

In Sect. 2, we present a review of the path integral representation of the relativistic particle propagator, and we unify the treatment for spin 0 and 1/2. For the latter case, we use the global projection [24]. In Sect. 3, by integration over spin variables we obtain an explicit expression for the Polyakov spin factor. The presence of the step gives rise to a correction of a delta potential in the spin–field coupling. Therefore, we calculate the Green function in the presence of the step corrected by this delta potential by using the

[a] e-mail: boudjedaa@gmail.com, boudjeda@caramail.com

perturbation technique. The corresponding positive- and negative-energy states are obtained. In Sect. 4 we calculate the pair-creation probability for the Dirac and the Klein–Gordon equations relative to this step potential. We give the results for Klein–Gordon particles as well as for Dirac ones, where the exclusion Pauli principle changes the normalization and, hence, the probability. Section 5 is devoted to the conclusion.

2 Path integral formulation

The path integral for the relativistic spinning particle propagator in interaction with an external electromagnetic field was proposed by Fradkin–Gitman [25, 26]. In this section, let us briefly recall the construction of this path integral representation. Multiplying the Dirac operator $(\hat{\not\pi} - m)$ by the matrix γ^5 ($\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$), the relative propagator will satisfy the following equation:

$$(\hat{\not\pi} - m\gamma^5)\tilde{S}(x,y) = -\delta^4(x-y)\,, \tag{1}$$

where $\tilde{S} = S^{\mathrm{c}}\gamma^5$, $\tilde{\gamma}^5 = \gamma^5$, $\tilde{\gamma}^\mu = \gamma^5\gamma^\mu$ and $[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_+ = 2\eta^{\mu\nu}$ with $\eta^{mn} = \mathrm{diag}(1,-1,-1,-1,-1)$ and $m, n = \overline{0,3}, 5$.

Formally, $\tilde{S}(x,y)$ is the matrix element in the coordinate space of the inverse Dirac operator, namely

$$\tilde{S}(x,y) = \langle x|\tilde{S}|y\rangle\,,$$

with

$$\tilde{S} = (\hat{\not\pi} - m\gamma^5)^{-1} = O^{-1}\,. \tag{2}$$

The operator $(\hat{\not\pi} - m\gamma^5)$ is a pure Fermi operator and its square is a Bose type operator that can be represented using the generalized Schwinger proper time representation:

$$\tilde{S}(x,y) = \int \mathrm{d}\lambda \int \mathrm{d}\chi \langle x| \exp\{\mathrm{i}[\lambda\tilde{O}^2 + \chi\tilde{O}]\}|y\rangle\,, \tag{3}$$

where χ is an odd variable anti-commuting with $\tilde{\gamma}$-matrices. This is known as a local projection. But there is another simple projection known as global [24], where the operator $\tilde{O}$ does not evolve, and its role is to project along global evolution the Klein–Gordon type states on the Dirac ones. To construct the related path integral representation, we define the propagator of the Dirac equation as a causal Green function, $\tilde{S}^{\mathrm{c}}(x,y)$, solution of the equation

$$(\hat{\not\pi} - m\gamma^5)\tilde{S}^{\mathrm{c}}(x,y) = -\delta^4(x-y)\,. \tag{4}$$

So

$$\tilde{S}^{\mathrm{c}}(x,y) = (\hat{\not\pi} - m\gamma^5)_{\mathrm{out}}\tilde{G}^{\mathrm{c}}(x,y)\,, \tag{5}$$

where

$$\hat{\not\pi} = \gamma^\mu\hat{\pi}_\mu \tag{6}$$

and $\tilde{G}^{\mathrm{c}}(x,y)$ is represented via the Schwinger proper time method as

$$\tilde{G}^{\mathrm{c}}(x,y) = -\mathrm{i}\int \mathrm{d}\lambda \langle x \mid \mathrm{e}^{-\mathrm{i}\hat{H}(\lambda)} \mid y\rangle\,. \tag{7}$$

To unify the study for spin 0 and 1/2, it is convenient to write the Hamiltonian $H(\lambda)$ subject to the electromagnetic interaction as follows:

$$\hat{H}_s = -\left(\hat{\pi}^2 - m^2 + \mathrm{i}se\left(\mathcal{F}_{\mu\upsilon}\tilde{\gamma}^\mu\tilde{\gamma}^\nu\right)^{(2s)}\right), \quad s = 0, \frac{1}{2}\,, \tag{8}$$

where we have adopted the following notation for any symbol A:

$$(A)^{(2s)} = \begin{cases} 1 & \text{for } s = 0\,, \\ A & \text{for } s = 1/2\,. \end{cases} \tag{9}$$

To present $\tilde{G}^{\mathrm{c}}_s(x,y)$ by means of a path integral we write, as usual, $\exp(-\mathrm{i}H(\lambda)) = [\exp(-\mathrm{i}H(\lambda)\varepsilon)]^N$, with $\varepsilon = 1/N$, and insert the N and $(N+1)$ resolutions of the identity, $\int |\,x\rangle\langle x\,|\,\mathrm{d}x = I$ and $\int |\,p\rangle\langle p\,|\,\mathrm{d}p = I$, between all the operators $\exp(-\mathrm{i}H(\lambda)\varepsilon)$. Introducing the N additional integrations, which ensure the reparametrization invariance of the theory, $\int \mathrm{d}\lambda_k\delta(\lambda_k - \lambda_{k-1}) = 1$, we get

$$\begin{aligned}\tilde{G}^{\mathrm{c}}_s(x,y) = {} & \lim_{N\to+\infty}(-\mathrm{i})\int \mathrm{d}\lambda_0 \int \mathrm{d}\lambda_1 \int \cdots \int \mathrm{d}\lambda_{N+1} \\ & \times \int \mathrm{d}x_1 \int \cdots \int \mathrm{d}x_N \prod_{k=1}^{N+1} \langle x_k \mid \mathrm{e}^{-\mathrm{i}\hat{H}_s(\lambda_k)\Delta\tau} \\ & \mid x_{k-1}\rangle\delta(\lambda_k - \lambda_{k-1})\,.\end{aligned} \tag{10}$$

For these δ-functions we use the integral representation

$$\delta(\lambda_k - \lambda_{k-1}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{e}^{\mathrm{i}\pi_\lambda(\lambda_k - \lambda_{k-1})}\,\mathrm{d}\pi_k \tag{11}$$

and replace the Dirac matrix by the Grassmannian trajectories following the functional identity:

$$\begin{aligned}& \mathcal{T}\exp\left\{\int_0^\lambda \rho_n(\tau)\tilde{\gamma}^n\,\mathrm{d}\tau\right\}|_{\rho=0} \\ & = \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\int_{\Psi(0)+\Psi(1)=\theta}\mathfrak{D}\Psi\exp \\ & \quad\times\left[\int_0^\lambda\left(\Psi_n\dot{\Psi}^n - 2\mathrm{i}\rho_n\Psi^n\right)\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right]\Bigg|_{\theta=0}^{\rho=0}\,,\end{aligned} \tag{12}$$

where the measure $\mathfrak{D}\Psi$ is given by

$$\mathfrak{D}\Psi = D\Psi\left[\int_{\Psi(0)+\Psi(1)=\theta} D\Psi\exp\left(\int_0^1 \Psi_n\dot{\Psi}^n\,\mathrm{d}\tau\right)\right]^{-1}\,, \tag{13}$$

and θ_μ and Ψ_μ are Grassmannian variables, anti-commuting with the γ-matrices. Finally inserting the result into (5), the causal Green function $\tilde{S}_s^c(x,y)$ will have the following reparametrized supersymmetric path integral representation:

$$\tilde{S}_s^c(x,y) = (-\mathrm{i})(\hat{\not\pi} - m\gamma^5)^{(2s)}_{\text{out}} \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta}{\delta\theta^n}\right) \int \mathrm{d}\lambda_0 \int Dx \times \int Dp \int D\lambda \int D\pi_\lambda \int_{\Psi(0)+\Psi(1)=\theta} \mathfrak{D}\Psi \times \exp\left\{\mathrm{i}\int_0^\lambda \left[\pi^2 - m^2 + 2\mathrm{i}se\,(\mathcal{F}_{\mu\nu}\Psi^\mu\Psi^\nu)^{(2s)} + p\dot{x} + \pi_\lambda\dot{\lambda} - \mathrm{i}\Psi_n\dot{\Psi}^n\right]\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right\}\Bigg|_{\theta=0}. \tag{14}$$

We call this form a global representation for the causal Green function. These path integral representations are applied successfully to solve many relativistic problems [9–16]. In the next section, we give an exact calculation of the Green function $\tilde{S}_s^c(x,y)$ for the case of the step potential.

3 Exact calculation

Let us now go to the exact calculation of the Dirac and Klein–Gordon Green function $G_s^c(x,y)$ relative to the step potential ($e\mathcal{A}_0 = eV_0\theta(x)$, $V_0 > 0$ and $\mathbf{A}$ =0), where

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{15}$$

The Green function $G_s^c(x,y)$ can be expressed as follows:

$$G_s^c(x,y) = \int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{-\mathrm{i}E(x-y)} R_s(\lambda) \times \tilde{\mathcal{G}}_E^c(x,y,\lambda), \tag{16}$$

where the function $\tilde{\mathcal{G}}_E^c(x,y,\lambda)$ is given only in terms of bosonic trajectories

$$\tilde{\mathcal{G}}_E^c(x,y,\lambda) = \int Dx^1 \int Dp^0 \int Dp^1 \times \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau \times \left[\frac{1}{2}(\dot{x}^1)^2 + \left(p^0 - \mathrm{e}V_0\theta(x)\right)^2 - ((p^1)^2 + m^2)\right]\right\}, \tag{17}$$

and the factor $R_s(\lambda)$ is given by

$$R_s(\lambda) = (-\mathrm{i}) \exp\left(\mathrm{i}\tilde{\gamma}^n \frac{\delta}{\delta\theta^n}\right) \int_{\Psi(0)+\Psi(1)=\theta} \mathfrak{D}\Psi \times \exp\left\{\left[2\mathrm{i}se\,(\mathcal{F}_{\mu\nu}\Psi^\mu\Psi^\nu)^{(2s)} - \mathrm{i}\Psi_n\dot{\Psi}^n\right]\mathrm{d}\tau + \mathrm{i}p^0T + \Psi_n(1)\Psi^n(0)\right\}\Big|_{\theta=0}. \tag{18}$$

First, we start by integrating $R_s(\lambda)$ over the Grassmann Ψ variables. In order to eliminate the boundary conditions $\Psi^n(1) + \Psi^n(0) = \theta^n$, we pass to velocity space after the variable change $\Psi^n(\tau)$ into the new variable $\omega^n(\tau)$ defined by

$$\Psi^n(\tau) = \frac{1}{2}\int_0^\lambda \varepsilon(\tau - \acute{\tau})\omega^n(\acute{\tau})\,\mathrm{d}\acute{\tau} + \frac{\theta^n}{2}. \tag{19}$$

Notice that the velocity $\omega(\tau)$ is a Grassmann variable and is now free from the boundary conditions. Moreover, following this transformation, we get

$$\int \mathfrak{D}\omega \exp\left\{-\frac{1}{2}\omega^n\varepsilon\omega_n + \frac{e}{2}\omega^\mu\varepsilon\mathcal{F}_{\mu\upsilon}\varepsilon\omega^\nu - e\theta^\mu\mathcal{F}_{\mu\upsilon}\varepsilon\omega^\nu - e\theta^\mu\theta^\nu\mathcal{F}_{\mu\upsilon}\right\} = \exp\left\{-\int_0^e \mathrm{d}\acute{e}\int_0^\lambda \mathrm{d}\tau\left(\Gamma^{10}\mathcal{F}_{10}\right) - \tanh\left(e\int_0^\lambda \mathrm{d}\tau\mathcal{F}_{10}(\tau)\right)\theta^1\theta^0\right\}, \tag{20}$$

where

$$I_\mu = -e\theta^\nu\mathcal{F}_{\upsilon\mu}\varepsilon, \quad \mathcal{L}_{\mu\nu} = \eta_{\mu\nu}\varepsilon - e\varepsilon\mathcal{F}_{\mu\upsilon}\varepsilon, \tag{21}$$

and

$$\begin{cases} \mathcal{L}_{\mu\nu}^{-1} & \text{is the inverse of } \mathcal{L}_{\mu\nu}, \\ \Gamma_{\mu\nu} = \frac{1}{2}\varepsilon\mathcal{L}_{\mu\nu}^{-1}\varepsilon. \end{cases} \tag{22}$$

Using the results of Appendix A, we get

$$R_s(\lambda) = (-\mathrm{i}) \exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\left[\cosh\left(eV_0\int_0^\lambda \delta(x)\,\mathrm{d}\tau\right) - \theta^1\,\theta^0\sinh\left(eV_0\int_0^\lambda \delta(x)\,\mathrm{d}\tau\right)\right]. \tag{23}$$

Applying the identity

$$\exp\left(\mathrm{i}\tilde{\gamma}^n\frac{\delta}{\delta\theta^n}\right)\theta^1\theta^0|_{\theta=0} = \frac{\delta}{\delta\xi^1}\frac{\delta}{\delta\xi^0}\exp\left(\mathrm{i}\tilde{\gamma}^n\xi_n\right)\Big|_{\xi=0} = -\tilde{\gamma}^1\tilde{\gamma}^0. \tag{24}$$

We thus find

$$G_s^c(x,y) = (-\mathrm{i})\int \mathrm{d}\lambda_0 \int Dx^1 \int \frac{\mathrm{d}p^0}{2\pi}\mathrm{e}^{\mathrm{i}p_0T}M(\lambda)\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2 - m^2\right)} \times \exp\left\{\mathrm{i}\int_0^\lambda\left[\frac{1}{2}(\dot{x}^1)^2 + eV_0(-2p_0 + eV_0\theta(x))\right]\mathrm{d}\tau\right\} \times \left[\cosh\left(eV_0\int_0^\lambda \delta(x)\,\mathrm{d}\tau\right) + \tilde{\gamma}^1\,\tilde{\gamma}^0\sinh\left(eV_0\int_0^\lambda \delta(x)\ \mathrm{d}\tau\right)\right]; \tag{25}$$

the measure $M(\lambda)$ is defined by

$$M(\lambda) = \int Dp^1\mathrm{e}^{-\mathrm{i}\int_0^\lambda (p^1)^2\,\mathrm{d}\tau}. \tag{26}$$

Writing

$$\cosh\left(eV_0\int_0^\lambda \delta(x)\,\mathrm{d}\tau\right),\sinh\left(eV_0\int_0^\lambda \delta(x)\,\mathrm{d}\tau\right)$$

in the exponential form, the Green function $\tilde{G}^{\mathrm{c}}(x,y)$ can be expressed as follows:

$$\tilde{G}^{\mathrm{c}}(x,y)=\tilde{G}^{+}+\tilde{G}^{-}+(\tilde{\gamma}^1\tilde{\gamma}^0)^{(2s)}(\tilde{G}^{+}-\tilde{G}^{-}), \quad (27)$$

where the related Green functions $\tilde{G}^{+}$ and $\tilde{G}^{-}$ are respectively defined by

$$\begin{aligned}\tilde{G}^{+}_s=&\frac{-\mathrm{i}}{2}\int\mathrm{d}\lambda_0\int Dx^1\int\frac{\mathrm{d}p_0}{2\pi}M(\lambda)\mathrm{e}^{\mathrm{i}p_0T}\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2-m^2\right)}\\&\times\exp\left[\mathrm{i}\int_0^\lambda\left\{\frac{1}{2}(\dot{x}^1)^2+eV_0\left(-2p_0+eV_0\theta(x)\right)\right.\right.\\&\left.\left.-\mathrm{i}seV_0\delta(x)\right\}\mathrm{d}\tau\right], \quad (28)\\ \tilde{G}^{-}_s=&\frac{-\mathrm{i}}{2}\int\mathrm{d}\lambda_0\int Dx^1\int\frac{\mathrm{d}p_0}{2\pi}M(\lambda)\mathrm{e}^{\mathrm{i}p_0T}\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2-m^2\right)}\\&\times\exp\left[\mathrm{i}\int_0^\lambda\left\{\frac{1}{2}(\dot{x}^1)^2+eV_0\left(-2p_0+eV_0\theta(x)\right)\right.\right.\\&\left.\left.+\mathrm{i}seV_0\delta(x)\right\}\,\mathrm{d}\tau\right]. \quad (29)\end{aligned}$$

The path integral expressions (28) and (29) correspond respectively to a physical system for which the Hamiltonian is given by

$$H=H_0\pm W, \quad (30)$$

and

$$\begin{aligned}H_0&=\frac{1}{2}(\dot{x}^1)^2+eV_0(-2p_0+\mathrm{e}V_0\theta(x))\\W&=\mathrm{e}sV_0\delta(x). \quad (31)\end{aligned}$$

This is a solvable path integral and has been already treated in non-relativistic quantum mechanics. In the general case, it is difficult to set up a direct calculation that yields an exact result for a general scalar potential and we can then proceed to the WKB approximation or an instanton technique. For our case, we turn to the perturbation method because of the presence of a delta distribution potential.

3.1 Perturbation method

As usually done [27], let us develop $\tilde{G}^{+}_s$ and $\tilde{G}^{-}_s$ in a perturbation series by expanding the exponential of $eV(x)$ as a power series; we obtain

$$\begin{aligned}\tilde{G}^{+}_s=&\frac{-\mathrm{i}}{2}\int\mathrm{d}\lambda_0\int Dx^1\int\frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0T}\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2-m^2\right)}\\&\times\left\{K(x_b,x_a,\lambda)+\sum_{n=1}^{\infty}(\mathrm{e}v_0)^n\int_0^\lambda\mathrm{d}\lambda_n\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{d}\lambda_1\right.\\&\times\prod_{J=1}^{N}\mathrm{d}x_jK(x_a,x_1)\delta(x_1)K(x_1,x_2)\\&\left.\times\delta(x_2)\cdots\delta(x_n)K(x_b,x_a)\right\}, \quad 0<\lambda_1<\lambda_2\cdots<\lambda,\end{aligned} \quad (32)$$

and

$$\begin{aligned}\tilde{G}^{-}_s=&\frac{-\mathrm{i}}{2}\int\mathrm{d}\lambda_0\int Dx^1\int\frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0T}\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2-m^2\right)}\\&\times\left\{K(x_b,x_a,\lambda)-\sum_{n=1}^{\infty}(ev_0)^n\int_0^\lambda\mathrm{d}\lambda_n\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{d}\lambda_1\right.\\&\times\prod_{J=1}^{N}\mathrm{d}x_jK(x_a,x_1)\delta(x_1)K(x_1,x_2)\\&\left.\times\delta(x_2)\cdots\delta(x_n)K(x_b,x_a)\right\}. \quad 0<\lambda_1<\lambda_2\cdots<\lambda,\end{aligned} \quad (33)$$

where $K(x_b,x_a,\lambda)$ is expressed by the following expression:

$$\begin{aligned}K(x_b,x_a,\lambda)=&\int Dx^1M(\lambda)\\&\times\exp\left[\mathrm{i}\int_0^\lambda\left(\frac{(\dot{x}^1)^2}{2}+eV_0(-2p_0+eV_0)\theta(x)\right)\,\mathrm{d}\tau\right].\end{aligned} \quad (34)$$

Let us remark that

$$-\mathrm{i}\int_0^{+\infty}\mathrm{d}\lambda_0\mathrm{e}^{\mathrm{i}\lambda\left(p_0^2-m^2\right)}K(x_b,x_a,\lambda)=g^{\mathrm{KG}}\left(x_b,x_a,p^0\right), \quad (35)$$

where $g^{\mathrm{KG}}\left(x_b,x_a,p^0\right)$ is the Klein–Gordon Green function relative to the step potential; for example for $E>m$ we have

$$g^{\mathrm{KG}}\left(x_b,x_a,p^0\right)=\begin{cases}\frac{1}{r_1}\left[\mathrm{e}^{\mathrm{i}r_1|x_b-x_a|}+\frac{r_1-r_2}{r_1+r_2}\mathrm{e}^{-\mathrm{i}r_1(x_b+x_a)}\right],\\ \quad x_a<0,\ x_b<0,\\ \frac{1}{r_1+r_2}\mathrm{e}^{\mathrm{i}(r_2x_b-r_1x_a)},\\ \quad x_a<0,\ x_b>0,\\ \frac{1}{r_1+r_2}\mathrm{e}^{-\mathrm{i}(r_1x_b-r_2x_a)},\\ \quad x_a>0,\ x_b<0,\\ \frac{1}{r_2}\left[\mathrm{e}^{\mathrm{i}r_2|x_b-x_a|}-\frac{r_1-r_2}{r_1+r_2}\mathrm{e}^{\mathrm{i}r_2(x_b+x_a)}\right],\\ \quad x_a>0,\ x_b>0,\end{cases} \quad (36)$$

and

$$r_1 = \sqrt{E^2 - m^2}\,,$$
$$r_2 = \begin{cases} \sqrt{(E-eV_0)^2 - m^2} & \text{for } E - eV_0 > 0\,, \\ -\sqrt{(E-eV_0)^2 - m^2} & \text{for } E - eV_0 > 0\,. \end{cases} \tag{37}$$

Inserting (35) and (36) into (32) and (33), now following the perturbation method [28–30], we get the following Green function form:

$$\tilde{G}_s^+ = \frac{1}{2}\int \frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0 T}\left\{g^{\mathrm{KG}}\left(x_b, x_a, p^0\right) + (\mathrm{e}sV_0)\frac{g^{\mathrm{KG}}\left(x_b, 0, p^0\right) g^{\mathrm{KG}}\left(0, x_a, p^0\right)}{1 - g^{\mathrm{KG}}(0,0,p^0)(eV_0)}\right\}, \tag{38}$$

and

$$\tilde{G}_s^- = \frac{1}{2}\int \frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0 T}\left\{g^{\mathrm{KG}}\left(x_b, x_a, p^0\right) - (\mathrm{e}sV_0)\frac{g^{\mathrm{KG}}\left(x_b, 0, p^0\right) g^{\mathrm{KG}}\left(0, x_a, p^0\right)}{1 + g^{\mathrm{KG}}(0,0,p^0)(eV_0)}\right\}. \tag{39}$$

Let us choose Δ^+ and Δ^- as follows:

$$\Delta_s^+ = \tilde{G}_s^+ + \tilde{G}_s^-\,,$$
$$\Delta_s^- = \tilde{G}_s^+ - \tilde{G}_s^-\,. \tag{40}$$

This implies

$$\Delta_s^+ = \int \frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0 T} g^{\mathrm{KG}}\left(x_b, x_a, p^0\right)\frac{(r_1+r_2)^2}{(r_1+r_2)^2 - (\mathrm{e}sV_0)^2}\,,$$
$$\Delta_s^- = \int \frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0 T} g^{\mathrm{KG}}\left(x_b, x_a, p^0\right)\frac{(\mathrm{e}sV_0)(r_1+r_2)}{(r_1+r_2)^2 - (\mathrm{e}sV_0)^2}\,. \tag{41}$$

Inserting these results in (27), we have

$$\tilde{G}_s^{\mathrm{c}}(x,y) = \tilde{G}_s^{\mathrm{c}}(x_b, x_a, T) = \Delta_s^+ + (\tilde{\gamma}^1\tilde{\gamma}^0)^{(2s)}\Delta_s^-\,. \tag{42}$$

The result of $\tilde{S}_s^{\mathrm{c}}(x,y)$ will then be

$$\tilde{S}_s^{\mathrm{c}}(x_b, x_a, T) = \left\{\mathrm{i}\tilde{\gamma}^0\frac{\partial}{\partial T} + \mathrm{i}\tilde{\gamma}^1\frac{\partial}{\partial x_b} - \tilde{\gamma}^0 eV_0\theta(x_b) - m\gamma^5\right\}^{(2s)} \times \int \frac{\mathrm{d}p_0}{2\pi}\mathrm{e}^{\mathrm{i}p_0 T}\tilde{g}_s^{\mathrm{KG}}\left(x_b, x_a, p^0\right)\,, \tag{43}$$

where

$$\tilde{g}_s^{\mathrm{KG}}\left(x_b, x_a, p^0\right) = \frac{(r_1+r_2)}{(r_1+r_2)^2 - (\mathrm{e}sV_0)^2} \times\left[(r_1+r_2) + \left(\tilde{\gamma}^1\tilde{\gamma}^0 eV_0\right)^{(2s)}\right] \times g^{\mathrm{KG}}\left(x_b, x_a, p^0\right)\,. \tag{44}$$

In order to determine the wave functions, it is convenient to write $\tilde{g}^{\mathrm{KG}}(x_b, x_a, p^0)$ in the integral form in which the conditions on E, x_b and x_a become implicit. That can be done using the residue theorem [31, 32]:

$$\tilde{g}_s^{\mathrm{KG}}\left(x_b, x_a, p^0\right) = \frac{1}{2\pi\mathrm{i}}\oint_{(C)} \mathrm{d}k\frac{\tilde{g}_s^{\mathrm{KG}}(x_b, x_a, k, E_k)}{P_0^2 - E_k^2}\,, \tag{45}$$

where (C) stands for a closed contour in the complex plane k, and $\tilde{g}^{\mathrm{KG}}(x_b, x_a, k, E_k)$ is given by

$$\begin{aligned}\tilde{g}_s^{\mathrm{KG}}&(x_b, x_a, k, E_k) \\ &= \frac{(k+k')}{(k+k')^2 - (\mathrm{e}sV_0)^2}\left[(k+k') + \left(\tilde{\gamma}^1\tilde{\gamma}^0 eV_0\right)^{(2s)}\right] \\ &\times\left\{\theta(x_b)\theta(-x_a)\frac{2k}{k+k'}\mathrm{e}^{\mathrm{i}k'x_b - \mathrm{i}kx_a}\right. \\ &+\theta(-x_b)\theta(x_a)\frac{2k}{k+k'}\mathrm{e}^{\mathrm{i}k'x_a - \mathrm{i}kx_b} \\ &+\theta(x_b)\theta(x_a)\left(\frac{k}{k'}\mathrm{e}^{-\mathrm{i}k'(x_b - x_a)} - \frac{k}{k'}\frac{k-k'}{k+k'}\mathrm{e}^{\mathrm{i}k'(x_b+x_a)}\right) \\ &\left.+\theta(-x_b)\theta(-x_a)\left(\mathrm{e}^{\mathrm{i}k(x_b - x_a)} + \frac{k}{k'}\frac{k-k'}{k+k'}\mathrm{e}^{-\mathrm{i}k(x_b+x_a)}\right)\right\}.\end{aligned} \tag{46}$$

In this case, we have (see (46))

$$E_k = \sqrt{k^2 + m^2}\,,$$
$$k' = \begin{cases}\sqrt{(E_k - eV_0\theta(x_b))^2 - m^2}\,, & E_k - \mathrm{e}V_0 > 0\,, \\ -\sqrt{(E_k - eV_0\theta(x_b))^2 - m^2}\,, & E_k - \mathrm{e}V_0 < 0\,;\end{cases} \tag{47}$$

thus, $\tilde{S}^{\mathrm{c}}(x_b, x_a, T)$ will take the following form:

$$\begin{aligned}&\tilde{S}_s^{\mathrm{c}}(x_b, x_a, T) = \\ &\frac{1}{2\pi\mathrm{i}}\left\{\mathrm{i}\tilde{\gamma}^0\frac{\partial}{\partial T} + \mathrm{i}\tilde{\gamma}^1\frac{\partial}{\partial x_b} - \tilde{\gamma}^0\mathrm{e}V_0\theta\left(x_b\right) - m\gamma^5\right\}^{(2s)} \\ &\times\oint_{(C)}\mathrm{d}k\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{e}^{\mathrm{i}p_0 T}}{P_0^2 - E_k^2}\times\left\{\theta(x_b)\theta(-x_a)\frac{2k}{k+k'}\mathrm{e}^{\mathrm{i}k'x_b - \mathrm{i}kx_a}\right. \\ &+\theta(-x_b)\theta(x_a)\frac{2k}{k+k'}\mathrm{e}^{\mathrm{i}k'x_a - \mathrm{i}kx_b} \\ &+\theta(x_b)\theta(x_a)\left(\frac{k}{k'}\mathrm{e}^{-\mathrm{i}k'(x_b - x_a)} - \frac{k}{k}\frac{k-k'}{k+k'}\mathrm{e}^{\mathrm{i}k'(x_b+x_a)}\right) \\ &\left.+\theta(-x_b)\theta(-x_a)\left(\mathrm{e}^{\mathrm{i}k(x_b - x_a)} + \frac{k}{k'}\frac{k-k'}{k+k'}\mathrm{e}^{-\mathrm{i}k(x_b+x_a)}\right)\right\}.\end{aligned} \tag{48}$$

Let us change p_0 by $-p_0$ and integrate by treating separately the cases $T > 0$ and $T < 0$. Considering the integral of the contour in the complex plane E_k, one obtains

$$\oint_{(C)}\frac{\mathrm{d}p_0}{2E_k}\mathrm{e}^{-\mathrm{i}p_0 T}\left[\frac{1}{p_0 - E_k} - \frac{1}{p_0 + E_k}\right] = 2\pi\mathrm{i}(\mathrm{Sgn}_{(C)})\sum\mathrm{Res.}\,, \tag{49}$$

where (C) is a contour and $E_k = \sqrt{k^2 + m^2 - \mathrm{i}\varepsilon}$. Then after integration we take the limit $\varepsilon \longrightarrow 0^+$ and in order to get the expression of the Green function $\tilde{S}_s^{\mathrm{c}}(x_b, x_a, T)$, it is convenient to introduce the energy variable instead of the momentum variable k via the change of variable $E_k = \sqrt{k^2 + m^2}$. Then after straightforward calculations and by applying the following transformation:

$$\tilde{\gamma}^\mu = \gamma^5 \gamma^\mu \,,$$
$$(-1)^{(2s)} S_s^{\mathrm{c}}(x_b, x_a, T) = \tilde{S}_s^{\mathrm{c}}(x, y)(\gamma^5)^{(2s)} \,, \qquad (50)$$

we obtain $S_s^{\mathrm{c}}(x_b, x_a, T)$ in the following matrix form:

$$\begin{aligned} S_s^{\mathrm{c}}(x_b, x_a, T) = \frac{1}{2\pi} \Bigg[\Bigg\{ \int_m^{eV_0 - m} \frac{\mathrm{d}E_k}{k} \\ \times \left[\theta(T) \mathrm{e}^{-\mathrm{i}E_k T} g_{1(s)+}^{\mathrm{KG}} + \theta(-T) \mathrm{e}^{\mathrm{i}E_k T} g_{1(s)-}^{\mathrm{KG}} \right] \Bigg\} \\ + \Bigg\{ \int_{eV_0 - m}^{eV_0 + m} \frac{\mathrm{d}E_k}{k} \\ \times \left[\theta(T) \mathrm{e}^{-\mathrm{i}E_k T} g_{2(s)+}^{\mathrm{KG}} + \theta(-T) \mathrm{e}^{\mathrm{i}E_k T} g_{2(s)-}^{\mathrm{KG}} \right] \Bigg\} \\ + \Bigg\{ \int_{eV_0 + m}^{+\infty} \frac{\mathrm{d}E_k}{k} \\ \times \left[\theta(T) \mathrm{e}^{-\mathrm{i}E_k T} g_{3(s)+}^{\mathrm{KG}} + \theta(-T) \mathrm{e}^{\mathrm{i}E_k T} g_{3(s)-}^{\mathrm{KG}} \right] \Bigg\} \Bigg], \end{aligned} \qquad (51)$$

where

$$\begin{aligned} g_{1+}^{\mathrm{KG}} = \left[(k + k') + (\sigma^1)^{(2s)} \mathrm{e}V_0 \right] \\ \times \left(\Psi_{1(s)\longrightarrow}^{(+)} \bar{\Psi}_{1(s)\longrightarrow}^{(+)} - \Psi_{1(s)\longleftarrow}^{(+)} \bar{\Psi}_{1(s)\longleftarrow}^{(+)} \right) , \\ g_{1-}^{\mathrm{KG}} = \left[(k + k') + (\sigma^1)^{(2s)} \mathrm{e}V_0 \right] \\ \times \left(\Psi_{1(s)\longrightarrow}^{(-)} \bar{\Psi}_{1(s)\longrightarrow}^{(-)} - \Psi_{1(s)\longleftarrow}^{(-)} \bar{\Psi}_{1(s)\longleftarrow}^{(-)} \right) . \end{aligned} \qquad (52)$$

g_{2+}^{KG} and g_{3+}^{KG} have the same form as g_{1+}^{KG}, and g_{2-}^{KG} and g_{3-}^{KG} have the same form as g_{1-}^{KG} with k' taking the following values:

$$k' = \begin{cases} \sqrt{(E_k - eV_0\theta(x_b))^2 - m^2} & \text{for } k \in [0, k_2] \,, \\ \mathrm{i}\sqrt{m^2 - (E_k - eV_0\theta(x_b))^2} & \text{for } k \in [k_2, k_1] \,, \\ -\sqrt{(E_k - eV_0\theta(x_b))^2 - m^2} & \text{for } k \in [k_1, +\infty] \,. \end{cases} \qquad (53)$$

Let us take into account the different k' values to obtain the general form of the wave function.

1. First, we have the left-propagating case with positive energy:

$$\begin{aligned} \Psi_{s\longrightarrow}^{(+)}(x) = \frac{(k + k')}{(k + k')^2 - (\mathrm{e}sV_0)^2} \\ [\theta(-x)(\mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx}) + \theta(x) T \mathrm{e}^{-\mathrm{i}k'x}] U^{(2s)}(x) \,. \end{aligned} \qquad (54)$$

2. Second, we have the right-propagating case with positive energy:

$$\begin{aligned} \Psi_{s\longleftarrow}^{(+)}(x) = \frac{(k + k')}{(k + k')^2 - (\mathrm{e}sV_0)^2} \\ \times \left[\theta(x) \sqrt{\frac{k}{k'}} (\mathrm{e}^{\mathrm{i}k'x} - R\mathrm{e}^{-\mathrm{i}k'x}) - \theta(-x) T \mathrm{e}^{-\mathrm{i}kx} \right] U^{(2s)}(x) \,. \end{aligned} \qquad (55)$$

3. Next the left-propagating case with negative energy:

$$\begin{aligned} \Psi_{s\longrightarrow}^{(-)}(x) = \frac{(k + k')}{(k + k')^2 - (\mathrm{e}sV_0)^2} [\theta(-x)(\mathrm{e}^{-\mathrm{i}kx} + R^* \mathrm{e}^{\mathrm{i}kx}) \\ + \theta(x) T^* \mathrm{e}^{\mathrm{i}k'x}] V^{(2s)}(x) \,. \end{aligned} \qquad (56)$$

4. Finally, we have the right-propagating case with negative energy:

$$\begin{aligned} \Psi_{s\longleftarrow}^{(-)} = \frac{(k + k')}{(k + k')^2 - (\mathrm{e}sV_0)^2} \\ \times \left[\theta(x) \sqrt{\frac{k}{k'}} (\mathrm{e}^{-\mathrm{i}k'x} - R^* \mathrm{e}^{\mathrm{i}k'x}) - \theta(-x) T^* \mathrm{e}^{\mathrm{i}kx} \right] V^{(2s)}(x) \,, \end{aligned} \qquad (57)$$

with $R = \frac{k+k'}{k-k'}$, $T = \frac{2\sqrt{kk'}}{k-k'}$, $k = \sqrt{E_k^2 + m^2}$ and

$$\begin{aligned} U^{(2s)}(x) = \begin{pmatrix} \sqrt{E_k - eV_0\theta(x_b) + m} \\ \mathrm{i}\sqrt{E_k - \mathrm{e}V\theta(x_b) - m} \end{pmatrix}^{(2s)} , \\ V^{(2s)}(x) = \begin{pmatrix} -\mathrm{i}\sqrt{E_k - eV_0\theta(x_b) - m} \\ \sqrt{E_k - eV\theta(x_b) + m} \end{pmatrix}^{(2s)} . \end{aligned} \qquad (58)$$

The non-relativistic limit. Let us now go to the non-relativistic limit of the spin 0 Green function case. In the limit $s \to 0$, the Green function (46) transforms into the K–G Green function (36), where

$$E = \sqrt{r_1^2 + m^2} \quad \text{and} \quad (E - \mathrm{e}v_0) = \sqrt{r_2^2 + m^2} \qquad (59)$$

and for the non-relativistic limit, we develop (59)

$$\begin{cases} E = m^2 \sqrt{1 + \frac{r_1^2}{m^2}} \simeq m + \frac{r_1^2}{2m} \,, \\ (E - \mathrm{e}v_0) = m^2 \sqrt{1 + \frac{r_2^2}{m^2}} \simeq m + \frac{r_2^2}{2m} \,, \end{cases} \qquad (60)$$

and choose ε and ε' as

$$\begin{cases} \varepsilon = (E - m) \simeq \frac{r_1^2}{2m} \,, \\ \varepsilon' = (E - \mathrm{e}v_0) - m \simeq \frac{r_2^2}{2m} \,. \end{cases} \qquad (61)$$

Inserting (60) and (61) into (46) we obtain

$$\begin{aligned}G(x_b,x_a) &= \theta(-x_b)\theta(-x_a)\frac{1}{r_1}\left[\mathrm{e}^{\mathrm{i}r_1(x_b-x_a)}+\frac{r_1-r_2}{r_1+r_2}\mathrm{e}^{-\mathrm{i}r_1(x_b+x_a)}\right]\\ &+\theta(x_b)\theta(x_a)\frac{1}{r_2}\left[\mathrm{e}^{\mathrm{i}r_2(x_b-x_a)}-\frac{r_1-r_2}{r_1+r_2}\mathrm{e}^{\mathrm{i}r_2(x_b+x_a)}\right]\\ &+\theta(x_b)\theta(-x_a)\frac{1}{r_1+r_2}\left[\mathrm{e}^{\mathrm{i}(r_2x_b-r_1x_a)}\right]\\ &+\theta(-x_b)\theta(x_a)\frac{1}{r_1+r_2}\left[\mathrm{e}^{-\mathrm{i}(r_1x_b-r_2x_a)}\right]. \end{aligned} \tag{62}$$

This is a non-relativistic expression limit for the Klein–Gordon Green function, and it is identical in form to that of the Schrödinger one [28–30].

4 Pair-creation probability

In this section, we calculate the pair-creation probability for relativistic particles subject to the step potential i.e. (15) for spin 0 and 1/2, using respectively the Klein–Gordon and Dirac Green functions $S_s^{\mathrm{c}}(x_b,x_a,T)$, calculated in the previous section.

4.1 Spin 0

Projecting the Klein–Gordon Green function $S_0^{\mathrm{c}}(x_b,x_a)$ on an outgoing particle and anti-particle states, we obtain the scalar pair-creation amplitude $A(k,k')$ as follows:

$$A(k,k')=\mathrm{i}\int_{\Sigma}\mathrm{d}\sigma\int_{\Sigma'}\mathrm{d}\sigma'\bar{\Psi}_k(x_b)\overleftrightarrow{\partial}_{x_b}S_0^{\mathrm{c}}(x_b,x_a)\overleftrightarrow{\partial}_{x_a}\Psi_{k'}(x_a)\,, \tag{63}$$

where $\Sigma(\Sigma')$ is the worldline of an observer of an (anti-) particle, $\mathrm{d}\sigma$ is the element of the hypersurface, and

$$\overleftrightarrow{\partial}=\overrightarrow{\partial}-\overleftarrow{\partial}\,. \tag{64}$$

We can choose the following wave functions Ψ_k and $\Psi_{k'}$ in the Klein–Gordon (spin 0) case [33, 34] in one spatial dimension relative to the step scalar potential (15):

$$\psi(x,t)=\mathrm{e}^{-\mathrm{i}Et}\begin{cases}\frac{1}{\sqrt{p}}\mathrm{e}^{\mathrm{i}kx}+r\frac{1}{\sqrt{p}}\mathrm{e}^{-\mathrm{i}kx}\,, & x\le 0\,,\\ s\frac{1}{\sqrt{p}}\mathrm{e}^{\mathrm{i}k'x}\psi_q\,, & x>0\,,\end{cases} \tag{65}$$

where $r=\frac{k-k'}{k+k'}$ and $s=\frac{2\sqrt{kk'}}{k+k'}$, with $k=\sqrt{E^2-m^2}$ and $k'=\sqrt{(E_k-eV_0)^2-m^2}$.

We now insert (65) and (51) into (63), and we obtain the amplitude for pair creation $A(k,k')$. Now $A(k,k')$ can be reduced to the following form:

$$A(k,k')=\delta(E-E')\delta^2(k-k')S(k)\,, \tag{66}$$

where the particle is in the state (E',k') and the anti-particle in the state (E,k); one can write down the one-pair-creation probability as follows:

$$\begin{aligned}P(k,k')&=\int|A(k,E,k',E')|^2\,\mathrm{d}E'\,\mathrm{d}k'\\ &=\int|\delta(E-E')\delta^2(k-k')S(k)|^2\,\mathrm{d}E'\,\mathrm{d}k'\,. \end{aligned}\tag{67}$$

Here the normalization is found by assuming that for each mode there exists an implicit delta function, which is a delta function squared, and one employs the usual trick [33, 34]:

$$\int\frac{\mathrm{d}\omega}{2\pi}[2\pi\delta(\omega-\omega')]^2=\int\mathrm{d}t\int\frac{\mathrm{d}\omega}{2\pi}2\pi\delta(\omega-\omega')\,, \tag{68}$$

and let us do the same thing for the second δ^2 in order to have the absolute probability for one-pair creation, since δ in (67) comes from energy conservation. As a result, we obtain the one-pair-creation probability $P(k,k')$ per unit time as follows:

$$\begin{aligned}P(k,k')&=|S(k)|^2\\ &=\frac{4kk'}{(k+k')^2}\,.\end{aligned} \tag{69}$$

As a result, we obtain the overall probability for pair creation and see that

$$P_0(k,k')=1-P(k,k')=\left(\frac{k-k'}{k+k'}\right)^2 \tag{70}$$

is the probability that the vacuum remains a vacuum in the presence of the high barrier potential.

We then observe that the expressions (69) and (70) are identical in form to those for their Klein–Gordon transmission and reflection coefficients. Now, we are ready to evaluate the related reflection and transmission coefficients and to show that by this path integral treatment of pair creation by the step potential the Klein paradox persists in the case of spin 0, if

a) $R+T=1$ for $eV_0+m<E_k<\infty$,
b) $R=1$ and $T=0$ for $eV_0-m<E_k<eV_0+m$,
c) $R-T=1$ for $m<E_k<eV_0-m$, which is the well-known Klein paradox.

We noted next that this result obtained with the Klein–Gordon expression is exactly the same as those obtained by [33–35] in the scalar step potential.

4.2 Spin 1/2

Projecting the spin 1/2 Green function $S_{1/2}^{\mathrm{c}}(x_b,x_a)$ on an outgoing particle and anti-particle state, we obtain the pair-creation amplitude $A(k,k')$, where the Pauli principle rigorously changes the normalization as follows:

$$A(k,k')=\mathrm{i}\int_{\Sigma}\mathrm{d}\sigma\int_{\Sigma'}\mathrm{d}\sigma'\bar{\Psi}_k(x_b)\gamma^0S_{1/2}^{\mathrm{c}}(x_b,x_a)\gamma^0\Psi_{k'}(x_a)\,, \tag{71}$$

where $\Sigma(\Sigma')$ is the worldline of an observer of an (anti-) particle, and $\mathrm{d}\sigma$ is the element of the hypersurface. We can choose the following wave functions Ψ_k and $\Psi_{k'}$ in the Dirac (spin 1/2) case [33, 34] in one spatial dimension relative to a step scalar potential (15):

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}Et}\begin{cases}\frac{1}{\sqrt{p}}\mathrm{e}^{\mathrm{i}kx}\psi_p + r\frac{1}{\sqrt{p}}\mathrm{e}^{-\mathrm{i}kx}\psi_{-p}\,, & x\le 0\,,\\ s\frac{1}{\sqrt{p}}\mathrm{e}^{\mathrm{i}k'x}\psi_q\,, & x>0\,,\end{cases} \tag{72}$$

where

$$r = \frac{\tilde{k}-\tilde{k}'}{\tilde{k}+\tilde{k}'}\,,\quad s = \frac{2\sqrt{\tilde{k}'\tilde{k}}}{\tilde{k}'+\tilde{k}}\,,$$
$$\tilde{k} = \frac{k}{E+m}\,,\quad \tilde{k} = \frac{k'}{E-eV_0+m}\,.$$

k and k' have the same form as in the scalar case and we defined

$$\psi_{\pm k} = \begin{pmatrix}\chi\\ \frac{\pm k}{E+m}\chi\end{pmatrix}\,,\quad \psi_{k'} = \begin{pmatrix}\chi\\ \frac{k'}{E-eV_0+m}\chi\end{pmatrix}\,. \tag{73}$$

The particle is in the state (E', k') and the anti-particle in the state (E, k). We now insert (72) and (51) for spin 1/2 into (71) and according to the same steps of calculating the probability for pair creation (spin 0), one will have the pair-production probability related to spin 1/2:

$$P(k,k') = |S(k)|^2 = \frac{4kk'}{(k+k')^2-(eV_0)^2}\,. \tag{74}$$

This last expression was obtained with the Dirac Green function coinciding exactly with the Dirac ones [36–38]. We observe that the expressions for the pair-creation probability for Dirac particles are identical in form to those for their Dirac transmission coefficient analogues and if we remove the effect of the spin 1/2 term (eV_0) from the above relation, (74), we find the reflection coefficient of the spin 0 case [33, 34]. We can also show that the Klein paradox persists by this treatment of pair creation in the spin 1/2 case and we can note that the same discussion as in the spin 0 case can be also obtained for the spin 1/2 particle; however, in the case of the Dirac particle, the density is a probability density and cannot be a charge density.

5 Conclusion

In this paper, we used the reparametrized supersymmetric formalism to calculate the Green function for the relativistic particle, with spin 0 and 1/2, in the presence of the step potential. This example is very illustrative but is also important in showing the properties of inhomogeneous fields. The calculations for spin 0 and 1/2 are unified. An explicit expression for the Polyakov spin factor is deduced. The Green functions are explicitly calculated and the presence of the step gives rise to a correction of the delta potential in the spin -field coupling. The Green function in the presence of the step corrected by this delta potential is calculated using the perturbation technique. The corresponding positive- and negative-energy states are obtained. The pair-creation probability in this potential is correctly calculated. This ensures that our first Green function calculations are correct. The method can be extended to more general solvable spatially inhomogeneous cases. For the latter, we can apply the WKB approximation or instanton techniques to have some information on the pair-creation phenomenon. This is under consideration.

Appendix : Inverse matrix $L^{-1}_{\mu\nu}(\tau,\acute{\tau})$

In the case of the step potential, $\mathcal{L}^{-1}_{\mu\nu}(\tau,\acute{\tau})$ and $\Gamma_{\alpha\beta}(\tau,\acute{\tau})$ are given by (22). We introduce a new matrix $\tilde{\Omega}_{\alpha\beta}(\tau,\acute{\tau})$ as follows:

$$\tilde{\Omega}_{\alpha\beta}(\tau,\acute{\tau}) = \int_0^\lambda \varepsilon(\tau-\tau_1)\mathcal{L}^{-1}_{\alpha\beta}(\tau_1,\acute{\tau})\,\mathrm{d}\tau_1\,. \tag{A.1}$$

To get the function $\Gamma_{\alpha\beta}(\tau,\acute{\tau})$ we write

$$\Gamma_{\alpha\beta}(\tau,\acute{\tau}) = \frac{1}{2}\int_0^\lambda \tilde{\Omega}_{\alpha\beta}(\tau,s)\varepsilon(s-\acute{\tau})\,\mathrm{d}s\,. \tag{A.2}$$

In our case

$$\mathcal{L}_{\mu\nu}(\tau,\acute{\tau}) = \varepsilon(\tau-\acute{\tau})\eta_{\mu\nu} - e\int_0^\lambda \varepsilon(\tau-\tau_1)\mathcal{F}_{\mu\nu}(\tau_1)\varepsilon(\tau_1-\acute{\tau})\,\mathrm{d}\tau_1\,. \tag{A.3}$$

Now we have

$$\int_0^\lambda \mathcal{L}_{\mu\nu}(\tau,s)(\mathcal{L}^{\nu\beta})^{-1}(s,\acute{\tau}) = \delta^\beta_\mu\delta(\tau-\acute{\tau})\,. \tag{A.4}$$

Therefore,

$$\int_0^\lambda \mathrm{d}s\Big[\varepsilon(\tau-s)\mathcal{L}^{-1}_{\mu\beta}(s,\acute{\tau}) - e\int_0^\lambda \mathrm{d}\tau_1\varepsilon(\tau-\tau_1)\mathcal{F}_{\mu\upsilon}(\tau_1)\varepsilon(\tau_1-s)\left(\mathcal{L}^\nu_\beta\right)^{-1}(s,\acute{\tau})\Big] = \eta_{\mu\beta}\delta(\tau-\acute{\tau})\,. \tag{A.5}$$

Substituting (A.1) into (A.5) we obtain

$$\tilde{\Omega}_{\mu\beta}(\tau,\acute{\tau}) - e\int_0^\lambda \mathrm{d}\tau_1\varepsilon(\tau-\tau_1)\mathcal{F}_{\mu\upsilon}(\tau_1)\tilde{\Omega}^\nu_\beta(\tau_1,\acute{\tau}) = \eta_{\mu\beta}\delta(\tau-\acute{\tau})\,. \tag{A.6a}$$

This equation is equivalent to the differential equation

$$\frac{\mathrm{d}\tilde{\Omega}_{\mu\beta}(\tau,\acute{\tau})}{\mathrm{d}\tau} - 2e\mathcal{F}_{\mu\upsilon}(\tau)\tilde{\Omega}^\nu_\beta(\tau,\acute{\tau}) = \eta_{\mu\beta}\frac{\mathrm{d}\delta(\tau-\acute{\tau})}{\mathrm{d}\tau}\,, \tag{A.6b}$$

with the initial condition

$$\tilde{\Omega}_{\mu\beta}(0,\acute{\tau})+e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}_{\mu v}(\tau_1)\tilde{\Omega}_{\beta}^{\nu}(\tau_1,\acute{\tau})=\eta_{\mu\beta}\delta(\acute{\tau})\,. \tag{A.7}$$

Inserting the general solution

$$\tilde{\Omega}(\tau,\acute{\tau})=\mathrm{e}^{2e\int_0^{\tau}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}\tilde{c}(\tau,\acute{\tau}) \tag{A.8}$$

into (A.6b), we get

$$\tilde{c}(s,\acute{\tau})=\int_0^{s}\mathrm{e}^{-2e\int_0^{\tau}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}\frac{\partial}{\partial\tau}\delta(\tau-\acute{\tau})\,\mathrm{d}\tau+\tilde{c}(\acute{\tau})\,. \tag{A.9}$$

In this way

$$\tilde{\Omega}(\tau,\acute{\tau})=\mathrm{e}^{2e\int_0^{\tau}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}[\tilde{c}(\acute{\tau})-\delta(\acute{\tau})]+\delta(\tau-\acute{\tau})$$
$$+2\mathrm{e}\mathcal{F}(\acute{\tau})\theta(\tau-\acute{\tau})\mathrm{e}^{-2e\int_0^{\acute{\tau}}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}\mathrm{e}^{2e\int_0^{\tau}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}, \tag{A.10}$$

and using the initial condition (A.7), taking into account $\tilde{\Omega}_{\mu\beta}(0,\acute{\tau})=\tilde{c}(\acute{\tau})$, we obtain

$$\tilde{c}(\acute{\tau})-\delta(\acute{\tau})=-e\frac{V^{-1}(\acute{\tau})V(\lambda)}{1+V(\lambda)}\mathcal{F}(\acute{\tau})\,, \tag{A.11}$$

where

$$V(\lambda)=\mathrm{e}^{2e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}\,, \tag{A.12}$$

$$V^{-1}(\lambda)=\mathrm{e}^{-2e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}(\tau_1)}\,. \tag{A.13}$$

Substituting (A.11) into (A.10), writing

$$\tilde{\Omega}(\tau,\acute{\tau})=\delta(\tau-\acute{\tau})+eV(\tau)$$
$$\times\left[\varepsilon(\tau-\acute{\tau})-\tanh\left(e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}(\tau_1)\right)\right]$$
$$\times V^{-1}(\acute{\tau})\mathcal{F}(\acute{\tau})\,, \tag{A.14}$$

and substituting (A.14) into (A.2), we obtain

$$\Gamma(\tau,\acute{\tau})=\frac{1}{2}V(\tau)\left[\varepsilon\,(\tau-\acute{\tau})-\tanh\left(e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}(\tau_1)\right)\right]$$
$$\times V^{-1}(\acute{\tau})\,. \tag{A.15}$$

According to (A.1), we get

$$\mathcal{L}^{-1}(\tau,\acute{\tau})=\frac{1}{2}\frac{\partial\tilde{\Omega}(\tau,\acute{\tau})}{\partial\tau}$$
$$=\varepsilon^{-1}(\tau,\acute{\tau})+\mathrm{e}^2\mathcal{F}(\tau)V(\tau)$$
$$\times\left[\varepsilon(\tau-\acute{\tau})-\tanh\left(e\int_0^{\lambda}\mathrm{d}\tau_1\mathcal{F}(\tau_1)\right)\right]$$
$$\times V^{-1}(\acute{\tau})\mathcal{F}(\acute{\tau})+eV(\tau)\delta(\tau-\acute{\tau})$$
$$\times V^{-1}(\acute{\tau})\mathcal{F}(\acute{\tau})\,. \tag{A.16}$$

References

1. J. Schwinger, Phys. Rev. **82**, 664 (1951)
2. A. Casher, H. Neuberger, S. Nussinov, Phys. Rev. D **20**, 179 (1979)
3. J.B. Hartle, S. Hawking, Phys. Rev. D **13**, 2188 (1976)
4. M.S. Marinov, V.S. Popov, Fortschr. Phys. **25**, 373 (1977)
5. I.H. Duru, N. Unal, Phys. Rev. D **34**, 959 (1986)
6. A.O. Barut, I.H. Duru, Phys. Rev. D **41**, 1312 (1990)
7. R.P. Feynman, Phys. Rev. **80**, 440 (1950)
8. R.P. Feynman, Phys. Rev. **84**, 108 (1951)
9. D.M. Gitman, S.I. Zlatev, W.D. Cruz, Brazil. J. Phys. **26**, 419 (1996)
10. D.M. Gitman, S.I. Zlatev, Phys. Rev. D **55**, 7701 (1997)
11. S. Zeggari, T. Boudjedaa, L. Chetouani, Czech. J. Phys. **51**, 185 (2001)
12. S. Zeggari, T. Boudjedaa, L. Chetouani, Phys. Scripta **64**, 285 (2001)
13. N. Boudiaf, T. Boudjedaa, L. Chetouani, Eur. Phys. J. **20**, 585 (2001)
14. N. Boudiaf, T. Boudjedaa, L. Chetouani, Eur. Phys. J. **22**, 593 (2001)
15. T. Boudjedaa, L. Chetouani, J. Phys. A **35**, 1651 (2002)
16. R. Rekioua, T. Boudjedaa, Eur. Phys. J. **49**, 1091 (2007)
17. G. Dunne, T. Hall, Phys. Rev. D **58**, 105 022 (1998)
18. H. Gies, K. Klingmuller, Phys. Rev. D **72**, 065 001 (2005)
19. E. Brezin, C. Itzykson, Phys. Rev. D **2**, 1191 (1970)
20. C. Itzykson, J. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980)
21. I.K. Affleck, O. Alvarez, N.S. Manton, Nucl. Phys. B **197**, 509 (1982)
22. G.V. Dunne, C. Schubert, Phys. Rev. D **72**, 105 004 (2005)
23. G.V. Dunne, Q.H. Wang, H. Gies, C. Schubert, Phys. Rev. D **73**, 065 028 (2006)
24. C. Alexandrou, R. Rosenfelder, A.W. Schreiber, Phys. Rev. A **59**, 1762 (1999)
25. E.S. Fradkin, D.M. Gitman, Phys. Rev. D **44**, 3230 (1991)
26. D.M. Gitman, A.V. Saa, Mod. Phys. Lett. A **8**, 463 (1993)
27. R.P. Feynman, A. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965)
28. S.V. Lawande, K.V. Bhagwat, Phys. Lett. A **131**, 8 (1988)
29. C. Grosche, Phys. Rev. Lett. **71**, 1 (1993)
30. C. Grosche, F. Steiner, Handbook of Feynman Path Integral (Springer, Berlin Heidelberg, 1998)
31. T. Boudjedaa, L. Chetouani, M. Merad, Il Nuovo Cimento B **114**, 1261 (1999)
32. T. Boudjedaa, L. Chetouani, M. Merad, Chin. J. Phys. **38**, 1019 (2000)
33. B.R. Holstein, Am. J. Phys. **66**, 507 (1998)
34. B.R. Holstein, Am. J. Phys. **67**, 499 (1999)
35. N. Dombey, A. Calogeracos, Phys. Rep. **315**, 41 (1999)
36. S. Flügge, Practical Quantum Mechanics (Springer, Berlin, 1994), pp. 213–219
37. N. Dombey, A. Calogeracos, Phys. Rep. **315**, 41 (1991)
38. W. Greiner, Relativistic Quantum Mechanics (Springer, Berlin, 1990), p. 35